

THE MEAN CURVATURE FLOW FOR ISOPARAMETRIC SUBMANIFOLDS

XIAOBO LIU* AND CHUU-LIAN TERNG†

ABSTRACT. A submanifold in space forms is *isoparametric* if the normal bundle is flat and principal curvatures along any parallel normal fields are constant. We study the mean curvature flow with initial data an isoparametric submanifold in Euclidean space and sphere. We show that the mean curvature flow preserves the isoparametric condition, develops singularities in finite time, and converges in finite time to a smooth submanifold of lower dimension. We also give a precise description of the collapsing.

1. INTRODUCTION

The mean curvature flow (abbreviated as MCF) of a submanifold $M \subset \mathbb{R}^N$ over an interval I is a map $f : I \times M \longrightarrow \mathbb{R}^N$ such that for all $t \in I$ and $x \in M$, $\frac{\partial}{\partial t} f(t, x)$ is equal to the mean curvature vector of $M(t) = f(t, M)$ at the point $x(t) = f(t, x)$. Mean curvature flows of convex hypersurfaces have been extensively studied in the literature (cf. [GH], [Hu]). An exposition of the work in this area was given in the book [Z]. Comparatively, the behavior of mean curvature flows of submanifolds with higher codimension is less understood (cf. [W]). This is partly due to the lack of understanding of collapsing and the formation of singularities of the flow equations in the higher codimensional case.

A submanifold M of a Riemannian manifold is *isoparametric* if its normal bundle is flat and principal curvatures along any parallel normal vector field are constant. The codimension of M is called the *rank* of M . An isoparametric submanifold M in \mathbb{R}^N is *full* if it is not contained in any proper hyperplane, and is *irreducible* if it is not a product of two isoparametric submanifolds. We refer to [T] for the basic properties and structure theories for isoparametric submanifolds. Principal orbits of isotropy representations of symmetric spaces are isoparametric, they are the only compact homogeneous isoparametric submanifolds in Euclidean space (cf. [PT]), and are called *generalized flag manifolds*. There are also infinite families of non-homogeneous isoparametric submanifolds which arise from representations of Clifford algebras (cf. [FKM]). All these non-homogeneous examples have

*Research was partially supported by NSF grant DMS-0505835.

†Research supported in part by NSF Grant DMS-0529756.

rank 2. A theorem of Thorbergsson [Th] asserts that compact full irreducible isoparametric submanifolds with rank bigger than 2 are always homogeneous.

A complete isoparametric submanifold of \mathbb{R}^N can be decomposed as the product of a compact, irreducible, isoparametric submanifold and a subspace of \mathbb{R}^N . Since mean curvature flows with affine subspaces of \mathbb{R}^N as initial data is trivial and the mean curvature flow starting from a product submanifold stays as product, we will only consider compact, full, irreducible isoparametric submanifolds.

Let M be an isoparametric submanifold of \mathbb{R}^N , and ξ a parallel normal vector field on M . Then $M_\xi = \{x + \xi(x) \mid x \in M\}$ is again a smooth submanifold (may have higher codimension), and the map $M \rightarrow M_\xi$ is either a diffeomorphism or a fibration with a generalized flag manifold as fiber. The family of these parallel sets forms a singular foliation of the ambient Euclidean space \mathbb{R}^N . Top dimensional leaves are all isoparametric in \mathbb{R}^N , and they are called *parallel isoparametric submanifolds*. Lower dimensional leaves are no longer isoparametric, and they are called *focal submanifolds*.

We show that if $f : M \times [0, T) \rightarrow \mathbb{R}^N$ is a solution of the MCF in \mathbb{R}^N with $f(\cdot, 0)$ isoparametric then $f(\cdot, t)$ is isoparametric for all $t \in [0, T)$, i.e., the MCF preserves isoparametric condition. This reduces the MCF to a system of ordinary differential equations. There is a Weyl group W associated to each isoparametric submanifold M that acts on the normal plane $p + \nu_p M$. The ODE given by the mean curvature flow with initial data an isoparametric submanifold is given by a vector field H smoothly defined on the interior of the Weyl chamber C of W but blows up at the boundary of C . However, we can use generators of W -invariant polynomials to change coordinate so that the vector field H becomes a polynomial vector field and its flows can be solved explicitly.

Every compact isoparametric submanifold is contained in a sphere. This sphere is also foliated by parallel isoparametric submanifolds and focal submanifolds. Each isoparametric foliation contains a unique isoparametric submanifold which is a minimal submanifold of this sphere. The mean curvature flow in \mathbb{R}^N with initial data a minimal submanifold in S^{N-1} behaves like the mean curvature flow of a sphere, i.e. it just shrinks homothetically along the radial direction and collapses to a point in finite time (cf. Lemma 3.8). If M is an isoparametric submanifold in \mathbb{R}^N which is not minimal in the sphere, then its mean curvature flow will converge to a focal submanifold F of positive dimension (cf. Corollary 3.9). In fact, M is a fibration over F with each fiber a homogeneous isoparametric submanifold of a lower dimensional Euclidean space. Each fiber of this fibration collapses to a point under the mean curvature flow in a finite time.

We summarize some of the main results of this paper in the following Theorem (cf. Theorem 3.5, Theorem 3.6, and Proposition 3.10):

Theorem 1.1. *The mean curvature flow in \mathbb{R}^N with initial data a compact isoparametric submanifold*

- (1) *converges to a focal submanifold in finite time T ,*
- (2) *if the fibration from the initial isoparametric submanifold to the limiting focal submanifold is a sphere fibration (this is the generic case), then the mean curvature flow $M(t)$ has type I singularity, i.e., there is a constant c_0 such that $\|\mathbf{II}(t)\|_\infty^2(T - t) \leq c_0$ for all $t \in [0, T)$, where $\|\mathbf{II}(t)\|_\infty$ is the sup norm of the second fundamental form of $M(t)$,*
- (3) *every focal submanifold is the limit of the mean curvature flow with some parallel isoparametric submanifold as initial data,*
- (4) *if M_1 and M_2 are distinct parallel full isoparametric submanifolds in \mathbb{R}^N that lie in the same sphere. Then the mean curvature flows in \mathbb{R}^N with initial data M_1 and M_2 collapse to two distinct focal submanifolds.*

The mean curvature flow in S^{N-1} with initial data an isoparametric submanifold behaves very similarly to the Euclidean mean curvature flow. In particular we have the following theorem:

Theorem 1.2. *Let M be an isoparametric submanifold of S^{N-1} . Then the mean curvature flow in S^{N-1} with M as initial data*

- (1) *is constant if M is minimal in S^{N-1} , or*
- (2) *converges to a focal submanifold of positive dimension in finite time if M is not minimal.*

An isometric action of G on a Riemannian manifold N is *polar* if there exists a closed embedded submanifold Σ of N that meets all G -orbits and meets orthogonally. Such Σ is called a *section* of the polar action. Principal orbits of a polar action in \mathbb{R}^n and S^n are isoparametric (cf. [PT]). We prove that if the G -action on N is polar then the mean curvature flow preserves G -orbits and the flow becomes an ordinary differential equation on the section Σ . We expect that methods developed in this paper can be applied to study mean curvature flows for orbits of polar actions with flat sections in symmetric spaces.

This paper is organized as follows: We give a brief review of properties of isoparametric submanifolds that are needed in section 2, present proofs of results stated in Theorem 1.1 in section 3, construct explicit solutions of the MCF in \mathbb{R}^N with initial data an isoparametric submanifold in section 4. Since focal submanifolds are smooth manifolds, we can consider their mean curvature flow. Most properties of the mean curvature flows for isoparametric submanifolds also hold for focal submanifolds. This will be briefly discussed in section 5. We describe MCF in spheres with initial data an isoparametric submanifold in spheres in section 6, and in the last section we discuss MCF in a Riemannian manifold N with initial data a principal orbit of a polar action on N .

The authors like to thank Mu-Tao Wang, Yng-Ing Lee, and Mao-Pei Tsui for discussions on types of singularities of the mean curvature flows.

2. PRELIMINARIES

Geometric and topological properties of isoparametric submanifolds can be found in [T]. In this section we briefly review the properties which will be used in this paper. Let $M \subset \mathbb{R}^N$ be a full compact isoparametric submanifold of rank k .

2.1. Curvature spheres and curvature normals.

The tangent bundle of M can be decomposed into orthogonal sums of *curvature distributions* $\{E_i \mid i = 1, \dots, g\}$ for some integer $g > 0$. At each point of M , E_i is a common eigenspace of the shape operators of M at that point. There are parallel normal vector fields \mathbf{n}_i such that the shape operator A_ξ has the property

$$A_\xi|_{E_i} = \langle \xi, \mathbf{n}_i \rangle \text{Id}_{E_i}$$

for all normal vector ξ . Each vector field \mathbf{n}_i is called the *curvature normal* of E_i . The rank of E_i is called the *multiplicity* of \mathbf{n}_i , which will be denoted by m_i . Each E_i is an integrable distribution whose leaves are m_i -dimensional round spheres with radius $1/\|\mathbf{n}_i\|$. Such spheres are called *curvature spheres*.

2.2. Weyl group associated to M .

For each $i \in \{1, \dots, g\}$, let $\sigma_i(x)$ be the antipodal point in the i -th curvature sphere passing through x . Then σ_i is an involution on M . The group W generated by $\sigma_1, \dots, \sigma_g$ is a crystallographic *Coxeter group*. It is known that M is irreducible if and only if W is irreducible. For each $x \in M$, W also acts as a reflection group on the affine normal space $x + \nu_x M$ generated by reflections along hyperplanes

$$L_i := \{x + \xi \mid \xi \in \nu_x M, 1 - \langle \xi, \mathbf{n}_i(x) \rangle = 0\}$$

for $i = 1, \dots, g$.

The intersection $\bigcap_{i=1}^g L_i$ consists of a single constant point which is denoted by a . Then M is contained in a sphere which is centered at a . Without loss of generality, we always **assume** that $a = 0$, i.e., M is contained in a sphere centered at the origin of \mathbb{R}^N . This condition is equivalent to

$$\langle -x, \mathbf{n}_i(x) \rangle = 1 \tag{2.1}$$

for all $x \in M$ and $i = 1, \dots, g$ (cf. [T, Corrolary 1.17]).

2.3. Parallel submanifold.

For any parallel normal vector field ξ on M , define

$$M_\xi := \{x + \xi(x) \mid x \in M\}.$$

If

$$1 - \langle \xi(x), \mathbf{n}_i(x) \rangle \neq 0 \tag{2.2}$$

for $i = 1, \dots, g$, then M_ξ is again an isoparametric submanifold with the same dimension as M . M_ξ is called the *parallel isoparametric submanifold* of M defined by ξ . The curvature normals of M_ξ at the point $x + \xi(x)$ are given by

$$\frac{\mathbf{n}_i(x)}{1 - \langle \xi(x), \mathbf{n}_i(x) \rangle}$$

with same multiplicities m_i for $i = 1, \dots, g$. The mean curvature vector of M_ξ at $x + \xi(x)$ is given by

$$H(x + \xi(x)) = \sum_{i=1}^g \frac{m_i \mathbf{n}_i(x)}{1 - \langle \xi(x), \mathbf{n}_i(x) \rangle}. \quad (2.3)$$

When condition (2.2) fails, M_ξ is still a smooth submanifold of \mathbb{R}^N , but it is no longer isoparametric. This submanifold is called a focal submanifold of M . The dimension of M_ξ is strictly smaller than that of M . The map

$$\begin{aligned} \pi : M &\longrightarrow M_\xi \\ x &\longmapsto x + \xi(x) \end{aligned}$$

is a fibration over M_ξ with each fiber an isoparametric submanifold in the normal space of M_ξ at $\pi(x)$. In fact, fix $x_0 \in M$, let C denote the Weyl chamber of W on $x_0 + \nu_{x_0}M$ containing x_0 , i.e.,

$$C = \{x_0 + \xi \mid \xi \in \nu_{x_0}M, \langle \xi, n_i \rangle < 1\}.$$

If $y_0 = x_0 + \xi(x_0)$ lies in the boundary of C and $y_0 \neq 0$, then the fiber of the fibration $M \rightarrow M_\xi$ is a generalized flag manifold with Weyl group W_{y_0} , the isotropy subgroup of W at y_0 .

For any parallel normal vector field ξ on M , the intersection of M_ξ with $x + \nu_x M$ is an orbit of W . In particular, if M_ξ is a parallel isoparametric submanifold, then it intersects each open Weyl chamber of W exactly once. Moreover, M_ξ is a focal submanifold if and only if $x + \xi(x)$ is contained in $\bigcup_{i=1}^g L_i$.

2.4. Isoparametric map and W -invariant polynomials.

Given a W -invariant polynomial f on $V = x_0 + \nu_{x_0}M$, there is a unique extension $\Psi(f)$ on \mathbb{R}^N such that $\Psi(f)$ is constant along any parallel submanifold M_ξ and $\Psi(f)|_V = f$. Moreover, $\Psi(f)$ is also a polynomial.

Let $\tilde{\Delta}$ and Δ denote the Laplacian in \mathbb{R}^N and V respectively. Then by Lemma 3.2 of [T],

$$F(x) = \tilde{\Delta}\Psi(f)(x) - \Psi(\Delta f)(x) = \sum_{i=1}^g \frac{m_i \langle \nabla f(x), \mathbf{n}_i \rangle}{\langle x, \mathbf{n}_i \rangle}. \quad (2.4)$$

is a polynomial on \mathbb{R}^N and is constant along parallel submanifolds of M . Moreover, if f is a homogeneous W -invariant polynomial of degree m on V , then F is a homogeneous polynomial of degree $m - 2$ on \mathbb{R}^N .

3. MEAN CURVATURE FLOWS FOR GENERAL ISOPARAMETRIC SUBMANIFOLDS

Let M be an isoparametric submanifold of \mathbb{R}^N , fix $x_0 \in M$, and W the Coxeter group associated to M . We prove that the MCF stays isoparametric and the MCF equation becomes a flow equation of a vector field H defined in the interior of the Weyl chamber of W containing x_0 in $x_0 + \nu_{x_0}M$ and the vector field H tends to infinity at the boundary of the Weyl chamber. We prove that solutions of the ODE $x' = H(x)$ only exists for finite time. To see the finer structure of the behavior of the blow-up of MCF, we use W -invariant polynomials to construct a new coordinate system for the Weyl chamber so that the corresponding vector field H becomes a polynomial vector field. We then analyse the behavior of flows of this polynomial vector field to obtain informations on the collapsing of the MCF.

Fix $x_0 \in M$. Since νM is globally flat, we can identify a vector $v \in \nu_{x_0}M$ and the unique parallel normal field \hat{v} along M defined by $\hat{v}(x_0) = v$. Let \mathbf{n}_i be curvature normals of M with multiplicity m_i for $i = 1, \dots, g$. We may view \mathbf{n}_i either as a global parallel normal vector field along M or an element in $\nu_{x_0}M$. The precise meaning should be clear from the context.

Let $\xi(t) \in \nu_{x_0}M$ be a one parameter family of normal vectors satisfying the flow equation

$$\dot{\xi}(t) = \sum_{i=1}^g \frac{m_i \mathbf{n}_i}{1 - \langle \xi(t), \mathbf{n}_i \rangle}, \quad \xi(0) = 0. \quad (3.1)$$

It follows from (2.3) that ξ is a solution of (3.1) if and only if the one parameter family of parallel submanifolds $M(t) := M_{\xi(t)}$ satisfy the *mean curvature flow* equation with $M(0) = M$. In other words, the MCF preserves the isoparametric condition:

Proposition 3.1. *If $f : M \times [0, T) \rightarrow \mathbb{R}^N$ satisfies the mean curvature flow in \mathbb{R}^N and $f(\cdot, 0)$ is isoparametric, then $f(\cdot, t)$ is isoparametric for all $t \in [0, T)$.*

Equation (3.1) does not make sense if $\langle \xi(t), \mathbf{n}_i \rangle = 1$ for some i . We will only study the flow equation under the condition

$$\langle \xi(t), \mathbf{n}_i \rangle < 1$$

for all $i = 1, \dots, g$. In other words, we require that $x_0 + \xi(t)$ stays in the same Weyl chamber as x_0 for all t . Under this condition, all $M(t)$ are still isoparametric.

Note that (3.1) is a system of non-linear ODE given by a vector field defined on the Weyl chamber C containing x_0 and the vector field blows up along the boundary of C . The study of MCF with isoparametric submanifolds as initial data reduces to the study of solutions of this ODE system.

Theorem 3.2. *Let $M \subset S^{N-1}(r_0)$ be an n -dimensional isoparametric submanifold in \mathbb{R}^N , and $x_0 \in M$. If $\xi(t)$ satisfies the mean curvature flow*

equation (3.1), then $x(t) = x_0 + \xi(t)$ satisfies

$$x'(t) = - \sum_{i=1}^g \frac{m_i \mathbf{n}_i}{\langle x(t), \mathbf{n}_i \rangle}, \quad (3.2)$$

with $x(0) = x_0$. Let $H(t)$ be the mean curvature vector of $M(t) = M_{\xi(t)}$ at the point $x(t)$. Then

- (a) $\langle x(t), H(t) \rangle = -n$,
- (b) $\|x(t)\|^2 = \|x(0)\|^2 - 2nt$.

Proof. By equation (2.1),

$$\langle x(t), \mathbf{n}_i \rangle = \langle x(0), \mathbf{n}_i \rangle + \langle \xi(t), \mathbf{n}_i \rangle = -1 + \langle \xi(t), \mathbf{n}_i \rangle$$

for all $i = 1, \dots, g$. Since

$$H(t) = - \sum_{i=1}^g \frac{m_i \mathbf{n}_i}{\langle x(t), \mathbf{n}_i \rangle}, \quad (3.3)$$

we have $\langle x(t), H(t) \rangle = - \sum_{i=1}^g m_i = -n$. This proves part (a). Part (b) follows from integrating the following formula

$$\frac{d}{dt} \|x(t)\|^2 = 2 \langle x(t), x'(t) \rangle = 2 \langle x(t), H(t) \rangle = -2n.$$

□

Hence we have

Corollary 3.3. *The mean curvature flow in \mathbb{R}^N with initial data an isoparametric submanifold in $S^{N-1}(r_0)$ exists only for finite time with maximal interval $[0, T)$, where $0 < T \leq T_0 = \frac{r_0^2}{2n}$.*

The following Theorem is the key in proving

- (1) the limits of two flows of (3.2) have two different limit on the boundary ∂C of the Weyl chamber C ,
- (2) every point of ∂C is a limit of some flow of (3.2).

Theorem 3.4. *Let M be a compact isoparametric submanifold in \mathbb{R}^N , W the Weyl group associated to M , $x_0 \in M$ a fixed point, and $V = x_0 + \nu_{x_0} M$. Let P_1, \dots, P_k be a set of generators of the ring $\mathbb{R}[V]^W$ of W -invariant polynomials on V such that P_i are homogeneous polynomials of degree s_i with $P_1(x) = \|x\|^2$ and $s_1 \leq s_2 \leq \dots \leq s_k$, and C the Weyl chamber of W containing x_0 in V . Let $P : \bar{C} \rightarrow \mathbb{R}^k$ be the map defined by $P(x) = (P_1(x), \dots, P_k(x))$. Then P is a homeomorphism from \bar{C} to a closed subset $B = P(\bar{C})$. Moreover, there is a polynomial map*

$$\eta = (\eta_1, \dots, \eta_k) : \mathbb{R}^k \rightarrow \mathbb{R}^k$$

with $\eta_1 = -2n$ and η_j is a polynomial in $\eta_1, \dots, \eta_{j-1}$ such that if $x : [0, T) \rightarrow C$ is a solution of (3.2), then $y(t) = P(x(t))$ is a solution of

$$y'(t) = \eta(y(t)) = (\eta_1(y(t)), \dots, \eta_k(y(t))).$$

Proof. Since each orbit of W intersect \overline{C} exactly once and the algebra of invariant polynomials separate W orbits, the map

$$\begin{aligned} P : \overline{C} \cap D(1) &\longrightarrow \mathbb{R}^k \\ x &\longmapsto (P_1(x), \dots, P_k(x)) \end{aligned}$$

is an injective continuous map. Since P is injective and proper, $B = P(\overline{C})$ is a closed subset of \mathbb{R}^k and P is a homeomorphism from \overline{C} to B .

The Coxeter group W acts on V . If f is a W -invariant homogeneous polynomial of degree j on V , then by equation (2.4),

$$F(x) := \sum_{i=1}^g m_i \frac{\langle \nabla f(x), \mathbf{n}_i \rangle}{\langle x, \mathbf{n}_i \rangle}$$

is a W -invariant homogenous polynomial of degree $j - 2$. Let $x(t)$ be a solution of (3.2), and $f(t) = f(x(t))$. By equation (3.3),

$$f'(t) = \langle \nabla f(x(t)), x'(t) \rangle = \langle \nabla f(x(t)), H(t) \rangle = -F(x(t)).$$

Hence $f'(t)$ is the value of a W -invariant polynomial of degree $k - 2$ evaluated at $x(t)$. In particular, $\frac{d^j}{dt^j} f(t) = 0$ if $j > k/2$. Therefore $f(t)$ is a polynomial in t .

Let $y(t) = (y_1(t), \dots, y_k(t)) = P(x(t))$, and

$$F_i(x) = \sum_{i=1}^g m_i \frac{\langle \nabla P_i(x), \mathbf{n}_i \rangle}{\langle x, \mathbf{n}_i \rangle}.$$

By equation (2.4), F_i is a W -invariant homogeneous polynomial on V of degree $s_i - 2$. Since $\mathbb{R}[V]^W = \mathbb{R}[P_1, \dots, P_k]$,

$$F_i = -\eta_i(P_1, \dots, P_{i-1})$$

for some polynomial η_i . But we have shown above that $y'_i(t) = -F_i(x(t))$, so

$$y'_i(t) = -F_i(x(t)) = \eta_i(y_1(t), \dots, y_{i-1}(t)).$$

This shows that $y(t)$ is an integral curve of the polynomial vector field η on \mathbb{R}^k . Since $y_1(t) = \|x(0)\|^2 - 2nt$, solution y can be solved explicitly by integrations. \square

The MCF equation (3.2) is given by the vector field

$$H(x) = - \sum_{i=1}^g \frac{m_i \mathbf{n}_i}{\langle x, \mathbf{n}_i \rangle},$$

which is smoothly defined on the Weyl chamber C of $x_0 + \nu_{x_0}M$ and blows up at the boundary ∂C . If we use generators of W -invariant polynomials on $x_0 + \nu_{x_0}M$ to change coordinates to P as in Theorem 3.4, then the vector field H becomes the polynomial vector field η on $P(C)$. Moreover, the flow of η can be solved explicitly and globally. Then apply P^{-1} to flows of η to get flows of (3.2).

Theorem 3.5. *For any compact isoparametric submanifold M in \mathbb{R}^N , the mean curvature flow always converges to a focal submanifold at a finite time. Moreover, if M_1 and M_2 are parallel full isoparametric submanifolds which are contained in the same sphere, then mean curvature flows with initial data M_1 and M_2 never intersect and they converge to two distinct focal submanifolds.*

Proof. We use the same notation as Theorem 3.4. Let $x : [0, T) \rightarrow C$ be the maximal interval for a solution of the mean curvature flow equation (3.2). Note that $P_i(t) = P_i(x(t))$ are well defined since $P_i(t)$ are polynomials in t . Therefore the mean curvature flow of $x_0 \in M$ must converge to $P^{-1}(P_1(T), \dots, P_k(T))$ which lies on the boundary of \overline{C} . The mean curvature flow of M then converges to the focal submanifold passing through this point.

We may assume that $x_i(0)$ lies in the unit sphere. Let T_i denote the maximum time for the solution $x_i(t)$. If $T_1 \neq T_2$, then $\|x_i(t)\|^2 = 1 - 2nt$, so $\lim_{t \rightarrow T_1^-} \|x_1(t)\|^2 \neq \lim_{t \rightarrow T_2^-} \|x_2(t)\|^2$. If $T_1 = T_2 = T$, then since $x_i(t)$ are solutions of (3.2) and $\langle x_i(t), n_j \rangle < 0$, we have

$$\frac{1}{2} \frac{d}{dt} \|x_1(t) - x_2(t)\|^2 = \sum_{i=1}^g m_i \frac{\langle x_1(t) - x_2(t), \mathbf{n}_i \rangle^2}{\langle x_1(t), n_i \rangle \langle x_2(t), \mathbf{n}_i \rangle} \geq 0. \quad (3.4)$$

This implies that $\|x_1(t) - x_2(t)\|^2$ increases in $t \in [0, T)$, hence

$$\lim_{t \rightarrow T^-} x_1(t) \neq \lim_{t \rightarrow T^-} x_2(t).$$

□

Theorem 3.6. *Every focal submanifold is a limit of the mean curvature flow of certain isoparametric submanifold.*

We need a couple Lemmas. First a simple Lemma on scaling and the proof is obvious:

Lemma 3.7. *If $f : M \times [0, T) \rightarrow \mathbb{R}^N$ is a solution to the mean curvature flow with $f(x, 0) = f_0(x)$, then given any $r \neq 0$, $\tilde{f}(x, t) = rf(x, r^{-2}t)$ is a solution with $\tilde{f}(x, 0) = rf_0(x)$.*

Lemma 3.8. *Let $f : M^n \rightarrow S^{N-1}(r_0)$ be an immersed minimal submanifold of a sphere with radius r_0 . For any $x \in M$, the solution to the mean curvature flow equation in \mathbb{R}^N with initial data M is given by*

$$F(x, t) = \sqrt{1 - (2nt/r_0^2)} f(x).$$

In particular, the mean curvature flow of M shrinks to a point homothetically in finite time $T_0 = r_0^2/(2n)$.

Proof. For minimal submanifolds of the sphere $S^{N-1}(r)$ with radius r , the mean curvature vector at a point x is $-nx/r^2$. Let $F(x, t) = r(t)f(x)$ for $x \in M$ with $r(t) \geq 0$. Then the mean curvature vector field of $F(\cdot, t)$ at

point x is given by $-\frac{n}{r_0^2 r(t)} f(x)$. So $F(x, t)$ satisfies the mean curvature flow equation for f if and only if

$$r'(t) = -n/(r_0^2 r(t)) \quad \text{and} \quad r(0) = 1.$$

It follows that $r(t) = \sqrt{1 - (2nt/r_0^2)}$. \square

Proof of Theorem 3.6

In each isoparametric family, there exists a unique isoparametric submanifold $M \subset S^{N-1}(1)$, which is minimal in $S^{N-1}(1)$. Let $x_0 \in M$. The mean curvature flow for minimal submanifold in spheres can be solved explicitly as in Lemma 3.8, i.e., $x(t) = \sqrt{1 - 2nt} x_0$ is a solution of (3.2) and $x(t) \in C$ for all $t \in [0, \frac{1}{2n}]$.

Recall that integral curves of $H(x) = -\sum_{i=1}^g \frac{m_i \mathbf{n}_i}{\langle x, \mathbf{n}_i \rangle}$ map to integral curves of the polynomial vector field η under the homeomorphism P defined in Theorem 3.4. Since the integral curve starting from x_0 lies in C , the flow of η starting at $P(x_0)$ lies in $P(C)$. But $-\eta$ is a polynomial vector field and the one-parameter subgroup ϕ_t generated by $-\eta$ is a globally defined polynomial map. So there exists $\delta > 0$ and an open subset \mathcal{U} of $P(\bar{C})$ such that \mathcal{U} contains the origin and $\phi_t(z) \in P(C)$ for $t \in (0, \delta)$ and $z \in \mathcal{U}$. This shows that the flow of $-\eta$ starting at the boundary of \mathcal{U} points inward in $P(C)$. Hence any boundary point of $P^{-1}(\mathcal{U})$ is a limit of some MCF with initial data in C . It follows from Lemma 3.7 that any focal submanifold can be a limit of some MCF with some initial isoparametric submanifold. \square

As consequence of Theorem 3.5 and Lemma 3.8, we have

Corollary 3.9. *Let M be an isoparametric submanifold. The mean curvature flow of M converges to a point if and only if it is minimal in the sphere containing it.*

Below we describe the rate of collapsing of the MCF for isoparametric submanifolds. Recall that a MCF, M_t , collapses at time $T < \infty$ is said to have *type I singularity* (cf. [W]) if there is a constant c_0 such that

$$\|\text{II}(t)\|_\infty^2 (T - t) \leq c_0$$

for all $t \in [0, T)$, where $\|\text{II}(t)\|_\infty$ is the sup norm of the second fundamental form for M_t .

Proposition 3.10. *Let $x(t)$ be a solution of the MCF (3.2), and T is the maximal time. Then*

- (1) $x(T) := \lim_{t \rightarrow T^-} x(t)$ exists and belong to the boundary ∂C of the Weyl chamber C ,
- (2) $\lim_{t \rightarrow T^-} \frac{\|x(t) - x(T)\|^2}{T - t} = 2m$, where $m = \dim(M_{x(0)}) - \dim(M_{x(T)})$,
- (3) if $x(T)$ lies in a highest dimensional stratum of ∂C , then the MCF has type I singularity.

Proof. We have proved (1) in Theorem 3.5. Statement (2) follows from the L'Hopital law:

$$\begin{aligned} \lim_{t \rightarrow T^-} \frac{\|x(t) - x(T)\|^2}{T - t} &= \lim_{t \rightarrow T^-} \frac{2\langle x(t) - x(T), x'(t) \rangle}{-1} \\ &= 2 \lim_{t \rightarrow T^-} \sum_{i=1}^g \langle x(t) - x(T), \frac{m_i \mathbf{n}_i}{\langle x(t), \mathbf{n}_i \rangle} \rangle \\ &= 2 \lim_{t \rightarrow T^-} \sum_{i \notin I} m_i \frac{\langle x(t) - x(T), \mathbf{n}_i \rangle}{\langle x(t), \mathbf{n}_i \rangle} + 2 \sum_{i \in I} m_i \frac{\langle x(t), \mathbf{n}_i \rangle}{\langle x(t), \mathbf{n}_i \rangle} = 2 \sum_{i \in I} m_i, \end{aligned}$$

which is the dimension of the fiber of $M_{x(0)} \rightarrow M_{x(T)}$. Here $I = \{1 \leq i \leq g \mid \langle x(T), \mathbf{n}_i \rangle = 0\}$.

We now prove statement (3). If $x(T)$ lies in a highest dimensional stratum of ∂C , then there exists a unique i such that $x(T)$ lies in the hyperplane defined by \mathbf{n}_i , i.e., $\langle x(T), \mathbf{n}_i \rangle = 0$. We may assume $i = 1$. Note that the norm square of the second fundamental form of $M_{x(t)}$ satisfies

$$\begin{aligned} \|\Pi(x(t))\|^2(T - t) &\leq \sum_{i=1}^g \frac{m_i \|\mathbf{n}_i\|^2}{\langle x(t), \mathbf{n}_i \rangle^2} (T - t) \\ &= \frac{m_1 \|\mathbf{n}_1\|^2 (T - t)}{\langle x(t), \mathbf{n}_1 \rangle^2} + \sum_{i=2}^g \frac{m_i \|\mathbf{n}_i\|^2}{\langle x(t), \mathbf{n}_i \rangle^2} (T - t). \end{aligned}$$

As $t \rightarrow T^-$, the second term tends to zero because $\langle x(T), \mathbf{n}_i \rangle \neq 0$ for all $i \geq 2$, and by the l'Hopital law the first term has the same limit as

$$\frac{-m_1 \|\mathbf{n}_1\|^2}{-2\langle x(t), \mathbf{n}_1 \rangle \sum_{i=1}^g m_i \frac{\langle \mathbf{n}_i, \mathbf{n}_1 \rangle}{\langle x(t), \mathbf{n}_i \rangle}}.$$

But the denominator tends to $-2m_1 \|\mathbf{n}_1\|^2$, so the limit is $1/2$. \square

We remark that there is an open dense subset \mathcal{O} of the Weyl chamber C such that the solution $x(t)$ of (3.2) with $x(0) \in \mathcal{O}$ converges to a point in a highest dimensional stratum of ∂C .

4. SOLUTIONS TO THE MEAN CURVATURE FLOW EQUATION

In this section, we use Theorem 3.4 to construct explicit solutions of the MCF (3.2) by selecting a set of generators P_1, \dots, P_k for the W -invariant polynomials and calculating flows of the polynomial vector field η .

We use the root system of the Coxeter group given in [GB]. Let M be a compact, irreducible isoparametric submanifold in \mathbb{R}^N , W its Weyl group, and \mathbf{n}_i its curvature normals. Let Π denote a set of simple roots of W , and Δ_+ the set of positive roots defined by Π . Then $\{\mathbb{R}\mathbf{n}_i \mid 1 \leq i \leq g\}$ is equal

to $\{\mathbb{R}\alpha \mid \alpha \in \Delta_+\}$. So the Weyl chamber C containing x_0 is precisely given by

$$C = \{x \in V \mid -\langle x, \alpha \rangle > 0 \text{ for all } \alpha \in \Pi\}.$$

The closure of C is

$$\overline{C} = \{x \in V \mid -\langle x, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Pi\},$$

and the MCF (3.2) becomes

$$x'(t) = - \sum_{\alpha \in \Delta^+} \frac{m_\alpha}{\langle x(t), \alpha \rangle} \alpha \quad (4.1)$$

where m_α is the multiplicity of the curvature normal which is parallel to α . Since (4.1) is invariant under re-scaling of each α , we may normalize roots of the Coxeter group to be of *unit length*.

If $M = M_1 \times M_2$ with M_i an isoparametric submanifold of \mathbb{R}^{N_i} for $i = 1, 2$, then the Weyl group of M is the product of the Weyl groups of M_1 and M_2 and the mean curvature flow of M is the product of the mean curvature flows of M_1 and M_2 . So without loss of generality, we may assume that M is an irreducible isoparametric submanifold. In the rest of this section, we work out explicit solutions for mean curvature flow equations for compact isoparametric submanifolds whose Coxeter group are A_k , B_k , D_k and G_2 .

Example 4.1. The A_k case

β

Suppose that $k \geq 2$. Let $\{e_1, \dots, e_{k+1}\}$ be the standard orthonormal basis of \mathbb{R}^{k+1} and (x_1, \dots, x_{k+1}) the corresponding coordinate. The set $\frac{1}{\sqrt{2}}(e_i - e_{i+1})$ with $1 \leq i \leq k$ is a simple root system of A_k , and the set of positive roots is $\frac{1}{\sqrt{2}}(e_i - e_j)$ with $1 \leq i < j \leq k+1$. Let

$$V := \{(x_1, \dots, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_1 + \dots + x_{k+1} = 0\}.$$

The normal space of an isoparametric submanifold of type A_k can be identified with V . The Coxeter group acts on V and is generated by all permutations of $\{e_1, \dots, e_{k+1}\}$. The open positive Weyl chamber C containing x_0 is

$$C = \{(x_1, \dots, x_{k+1}) \in V \mid x_1 < x_2 < \dots < x_{k+1}\}.$$

Since the multiplicities of curvature spheres are invariant under the action of the Coxeter group, there is only one possible multiplicity which we denote by m . So (4.1) is

$$x' = - \sum_{i < j} m \frac{e_i - e_j}{x_i - x_j}, \quad (4.2)$$

which implies that

$$\frac{1}{m} \frac{d}{dt} x_i = \sum_{j \neq i} \frac{1}{x_j - x_i}, \quad 1 \leq i \leq k+1. \quad (4.3)$$

If $x(0) \in V$, then $x(t) \in V$ for all t . This follows from the simple fact that $\frac{d}{dt}(x_1 + \dots + x_{k+1}) = 0$.

Let σ_r be the r -th elementary symmetric polynomial in x_1, \dots, x_{k+1} :

$$\sigma_r = \sum_{1 \leq i_1 < \dots < i_r \leq k+1} x_{i_1} \cdots x_{i_r},$$

and $\sigma_0 = 1$. Let $x(t)$ be a solution of (4.2), and

$$y_r(t) = \sigma_r(x(t)).$$

We claim that

$$\begin{cases} y'_2 = n, \\ y'_r = \frac{1}{2} m(k-r+3)(k-r+2)y_{r-2}. \end{cases} \quad (4.4)$$

To see this, we compute

$$\begin{aligned} \frac{r!}{m} \frac{d}{dt} y_r &= \frac{1}{m} \frac{d}{dt} \sum_{i_1 \neq \dots \neq i_r} x_{i_1} \cdots x_{i_r} \\ &= \sum_{i_1 \neq \dots \neq i_r} \sum_{q=1}^r x_{i_1} \cdots \widehat{x_{i_q}} \cdots x_{i_r} \sum_{j \neq i_q} \frac{1}{x_j - x_{i_q}} \end{aligned}$$

Write

$$\sum_{j \neq i_q} \frac{1}{x_j - x_{i_q}} = \sum_{j \neq i_1, \dots, i_r} \frac{1}{x_j - x_{i_q}} + \sum_{1 \leq p \leq r, p \neq q} \frac{1}{x_{i_p} - x_{i_q}}.$$

For fixed indices $i_1, \dots, \widehat{i_q}, \dots, i_r$,

$$\sum \frac{1}{x_j - x_{i_q}} = 0$$

if the summation is running over all possible values for i_q and j such that $i_q \neq j$ and both of them are not equal to $i_1, \dots, \widehat{i_q}, \dots, i_r$. Therefore

$$\frac{r!}{m} \frac{d}{dt} y_r = \sum_{i_1 \neq \dots \neq i_r} x_{i_1} \cdots x_{i_r} \sum_{1 \leq p, q \leq r, p \neq q} \frac{1}{x_{i_q}(x_{i_p} - x_{i_q})}. \quad (4.5)$$

Write

$$\sum_{1 \leq p, q \leq r, p \neq q} \frac{1}{x_{i_q}(x_{i_p} - x_{i_q})} = \sum_{p > q} \frac{1}{x_{i_q}(x_{i_p} - x_{i_q})} + \sum_{p < q} \frac{1}{x_{i_q}(x_{i_p} - x_{i_q})}.$$

Switch the indices p and q in the second term and adding the first term, we have

$$\begin{aligned} \sum_{1 \leq p, q \leq r, p \neq q} \frac{1}{x_{i_q}(x_{i_p} - x_{i_q})} &= \sum_{p > q} \frac{1}{x_{i_p} - x_{i_q}} \left(\frac{1}{x_{i_q}} - \frac{1}{x_{i_p}} \right) \\ &= \sum_{p > q} \frac{1}{x_{i_p} x_{i_q}}. \end{aligned}$$

Hence by equation (4.5)

$$\begin{aligned} \frac{r!}{m} \frac{d}{dt} y_r &= \sum_{i_1 \neq \dots \neq i_r} \sum_{1 \leq p, q \leq r, p > q} x_{i_1} \cdots \widehat{x_{i_q}} \cdots \widehat{x_{i_p}} \cdots x_{i_r} \\ &= \frac{1}{2} r(r-1)(k-r+3)(k-r+2)(r-2)! y_{r-2} \end{aligned}$$

This proves the claim.

The explicit formula for $y_r(t)$ can be obtained from (4.4) recursively, and it is a polynomial in t and initial conditions $x_1(0), \dots, x_{k+1}(0)$. For each t , we can obtain $x_1(t), \dots, x_{k+1}(t)$ as the $k+1$ solutions of the following polynomial equation in z :

$$\sum_{r=0}^{k+1} (-1)^{k+1-r} y_{k+1-r}(t) z^r = 0, \quad (4.6)$$

with the property

$$x_1(t) < x_2(t) < \dots < x_{k+1}(t).$$

Example 4.2. The B_k case

Suppose that $k \geq 2$. Let $\{e_1, \dots, e_k\}$ be the standard orthonormal basis of \mathbb{R}^k and (x_1, \dots, x_k) the corresponding coordinate. We identify \mathbb{R}^k with a normal space of an isoparametric submanifold of type B_k . The set e_k and $\frac{1}{\sqrt{2}}(e_i - e_{i+1})$ with $1 \leq i \leq k-1$ is a simple root system of B_k , and the set of positive roots are e_i with $1 \leq i \leq k$ and $\frac{1}{\sqrt{2}}(e_i \pm e_j)$ with $1 \leq i < j \leq k$. The Coxeter group is generated by all permutations and sign changes of $\{e_1, \dots, e_k\}$. Since the multiplicities of curvature spheres are invariant under the action of the Coxeter group, there are only two possible multiplicities. Let m_1 be the multiplicities of curvature spheres corresponding to $\frac{1}{\sqrt{2}}(e_i \pm e_j)$, and m_2 the multiplicities of curvature spheres corresponding to e_i . So the MCF (4.1) is

$$-x' = m_1 \left(\sum_{i < j} \frac{e_i + e_j}{x_i + x_j} + \frac{e_i - e_j}{x_i - x_j} \right) + m_2 \sum_i \frac{e_i}{x_i}.$$

So

$$-x'_i = \frac{m_2}{x_i} + m_1 \left(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j} \right).$$

Set $y_i = x_i^2$. Then we have

$$\frac{1}{2} \frac{d}{dt} y_i = -m_2 - m_1 \sum_{j \neq i} \frac{2y_i}{y_i - y_j}. \quad (4.7)$$

Let $s_0 = 1$, s_i the i -th elementary symmetric polynomial of y_1, \dots, y_k , and

$$\zeta_r(t) = s_r(x(t)).$$

We claim that

$$\zeta'_j = -2(k-j+1)(m_2 + m_1(k-j))\zeta_{j-1}, \quad 1 \leq j \leq k. \quad (4.8)$$

Note that when $j = 1$ the right hand side is $-2k(m_2 + m_1(k-1))$, which is equal to $-2n$. To prove this claim, we compute as follows: First

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (y_{i_1} \cdots y_{i_j}) &= \sum_{l=1}^j \frac{y_{i_1} \cdots y_{i_j}}{y_{i_l}} \left(-m_2 - 2m_1 \sum_{p \neq i_l} \frac{y_{i_l}}{y_{i_l} - y_p} \right) \\ &= y_{i_1} \cdots y_{i_j} \left(-m_2 \sum_{l=1}^j \frac{1}{y_{i_l}} - 2m_1 \sum_{l=1}^j \sum_{p \neq i_l} \frac{1}{y_{i_l} - y_p} \right). \end{aligned}$$

Since

$$\sum_{l=1}^j \sum_{p \neq i_l} \frac{1}{y_{i_l} - y_p} = \sum_{l=1}^j \sum_{1 \leq q \leq j, q \neq l} \frac{1}{y_{i_l} - y_{i_q}} + \sum_{l=1}^j \sum_{p \neq i_1, \dots, i_j} \frac{1}{y_{i_l} - y_p},$$

and the first term on the right hand side is 0, we have

$$\begin{aligned} \frac{d}{dt} s_j &= \frac{1}{j!} \frac{d}{dt} \left(\sum_{i_1 \neq \dots \neq i_j} y_{i_1} \cdots y_{i_j} \right) \\ &= -\frac{2m_2}{j!} \sum_{l=1}^j \sum_{i_1 \neq \dots \neq i_j} \frac{y_{i_1} \cdots y_{i_j}}{y_{i_l}} - \frac{4m_1}{j!} \sum_{l=1}^j \sum_{i_1 \neq \dots \neq i_j \neq p} \frac{y_{i_1} \cdots y_{i_j}}{y_{i_l} - y_p}. \quad (4.9) \end{aligned}$$

The first term on the right hand side of this equation is $-2m_2(k-j+1)\zeta_{j-1}$. To compute the second term, we notice that

$$\sum_{i_1 \neq \dots \neq i_j \neq p} \frac{y_{i_1} \cdots y_{i_j}}{y_{i_l} - y_p}$$

can be written as

$$\sum_{i_1 \neq \dots \neq i_j, i_l > p} \frac{(y_{i_1} \cdots \widehat{y_{i_l}} \cdots y_{i_j}) y_{i_l}}{y_{i_l} - y_p} + \sum_{i_1 \neq \dots \neq i_j, i_l < p} \frac{(y_{i_1} \cdots \widehat{y_{i_l}} \cdots y_{i_j}) y_{i_l}}{y_{i_l} - y_p}.$$

Switching the indices i_l and p in the second term and adding to the first term, we have

$$\sum_{i_1 \neq \dots \neq i_j \neq p} \frac{y_{i_1} \cdots y_{i_j}}{y_{i_l} - y_p} = \sum_{\substack{i_1 \neq \dots \neq i_j \neq p \\ i_l > p}} y_{i_1} \cdots \widehat{y_{i_l}} \cdots y_{i_j}$$

Therefore the second term on the right hand side of equation (4.9) is

$$-2m_1(k-j+1)(k-j)s_{j-1}.$$

This proves the claim.

Note that explicit formula for $\zeta_i(t)$ can be obtained from (4.8) recursively, and it is always a degree i polynomial in t and initial conditions $y_1(0), \dots, y_k(0)$. Let $y_1(t), \dots, y_k(t)$ be the k roots of

$$\sum_{r=0}^k (-1)^{k-r} s_{k-r}(t) z^r = 0, \quad (4.10)$$

with the property $y_1(t) > y_2(t) > \dots > y_k(t) > 0$. Then $x_i(t) = -\sqrt{y_i(t)}$ for $i = 1, \dots, k$.

Example 4.3. The D_k case

Suppose that $k \geq 4$. Let $\{e_1, \dots, e_k\}$ be the standard orthonormal basis of \mathbb{R}^k and (x_1, \dots, x_k) the corresponding coordinate. We identify \mathbb{R}^k with a normal space of an isoparametric submanifold of type D_k . The set of simple roots are $\frac{1}{\sqrt{2}}(e_{k-1} + e_k)$ and $\frac{1}{\sqrt{2}}(e_i - e_{i+1})$ with $1 \leq i \leq k-1$, and the set of positive roots is $\{\frac{1}{\sqrt{2}}(e_i \pm e_j) \mid 1 \leq i < j \leq k\}$. The Coxeter group is generated by all permutations and even number of sign changes of $\{e_1, \dots, e_k\}$. The open positive Weyl chamber C is

$$C := \{x \in \mathbb{R}^k \mid x_1 < x_2 < \dots < x_k \text{ and } x_{k-1} + x_k < 0\}.$$

All multiplicity for curvature spheres are equal and will be denoted by m . The mean curvature flow equation (4.1) becomes

$$-x' = m \sum_{i < j} \frac{e_i - e_j}{x_i - x_j} + \frac{e_i + e_j}{x_i + x_j}. \quad (4.11)$$

$$\frac{1}{m} \frac{d}{dt} x_i = \sum_{q \neq i} \frac{2x_i}{x_q^2 - x_i^2}.$$

Multiply both sides by $\frac{x_i}{2}$ to obtain

$$\frac{1}{4m} \frac{d}{dt} y_i = \sum_{q \neq i} \frac{y_i}{y_q - y_i} \quad (4.12)$$

where $y_i := x_i^2$ for all $i = 1, \dots, k$.

Let s_i be the i -th elementary symmetric polynomial of y_1, \dots, y_k and we set $s_0 = 1$. Then s_1, \dots, s_{k-1} and $\sqrt{s_k}$ generate the algebra of polynomials invariant under the action of the Coxeter group. Note that equation (4.12) is a special case of the equation (4.7) with $m_2 = 0$ and $m_1 = m$. Set $\zeta_j(t) = s_j(x(t))$. The proof of (4.8) also works here, and we obtain

$$\zeta'_r = -2m(k-r+1)(k-r)\zeta_{r-1}, \quad 1 \leq r \leq k.$$

Example 4.4. The rank 2 cases

We use a different set of generators for the ring of W -invariant polynomials to compute explicit solutions of the MCF (3.2) for the rank 2 cases. The Weyl group is the dihedral group with $2g$ elements. We identify the

normal space of a rank 2 isoparametric submanifold with $\mathbb{R}^2 = \mathbb{C}$, and use $e^{\frac{ik\pi}{g}}$ with $0 \leq k < g$ as positive roots. Then

$$P_1(x) = x_1^2 + x_2^2, \quad P_2(x) = \operatorname{Re}((x_1 + ix_2)^g)$$

form a set of generator of the ring of invariant polynomials ($g = 3$ for A_2 , $g = 4$ for B_2 , and $g = 6$ for G_2). It is known (cf. [PTb]) that

- (1) if $g = 3, 6$, then all multiplicities are equal and are either 1 or 2; and if $g = 4$, then there are two positive integers $m_1 \leq m_2$ such that the multiplicity corresponding to $\mathbb{R}\mathbf{n}_j = \mathbb{R}e^{\frac{j\pi i}{4}}$ is m_1 for j even and is m_2 for j odd.
- (2) Let $F_i = \sum_{i=1}^g \frac{m_i \langle \nabla P_i, \mathbf{n}_i \rangle}{\langle x, \mathbf{n}_i \rangle}$ for $i = 1, 2$ be as in Theorem 3.4. Then

$$F_1(x) = 2n, \quad F_2(x) = \begin{cases} 0 & \text{if } g = 3 \text{ or } 6, \\ 8(m_2 - m_1)(x_1^2 + x_2^2), & \text{if } g = 4, \end{cases}$$

If $x(t)$ is a solution of the MCF, then it follows from Theorem 3.4 that $y(t) = (P_1(x(t)), P_2(x(t)))$ satisfies

$$\begin{cases} y_1' = -2n, \\ y_2' = \begin{cases} 0, & \text{if } g = 3, 6, \\ -8(m_2 - m_1)y_1, & \text{if } g = 4. \end{cases} \end{cases} \quad (4.13)$$

By Lemma 3.7, it suffices to consider initial data $x_0 = e^{i\phi_0}$ for the MCF. The corresponding solution for (4.13) with initial data $y(0) = (1, \cos g\phi_0)$ is

$$\begin{aligned} y_1(t) &= 1 - 2nt, \quad y_2(t) = \cos(g\phi_0), \quad \text{if } g = 3, 6, \\ y_1(t) &= 1 - 2nt, \quad y_2(t) = \cos 4\phi_0 - 8(m_2 - m_1)(t - nt^2), \quad \text{if } g = 4. \end{aligned}$$

Set $x(t) = r(t)e^{i\phi(t)}$. Since $y_1(t) = r^2(t)$ and $y_2(t) = r^g(t) \cos g\phi(t)$, solution of the MCF for rank 2 case with initial data $e^{i\phi_0}$ is $r(t) = (1 - 2nt)^{\frac{1}{2}}$ and

$$\phi(t) = \begin{cases} \frac{1}{g} \cos^{-1} \left(\frac{\cos g\phi_0}{(1-2nt)^{\frac{g}{2}}} \right), & g = 3, 6, \\ \frac{1}{4} \cos^{-1} \left(\frac{\cos 4\phi_0 - 8(m_2 - m_1)(t - nt^2)}{(1-2nt)^2} \right), & g = 4. \end{cases}$$

We claim that the isoparametric submanifold through $x_0 = e^{i\phi_0}$ is minimal in sphere, where

$$\phi_g = \begin{cases} \frac{\pi}{2g}, & g = 3, 6, \\ \frac{1}{4} \cos^{-1} \left(\frac{m_2 - m_1}{m_2 + m_1} \right), & g = 4. \end{cases} \quad (4.14)$$

To see this, by Lemma 3.8, the the polar angle of the MCF flow starting at a minimal submanifold in sphere must be some constant ϕ_0 . So:

- (i) For $g = 3, 6$, we have $y_2(t) = r(t)^g \cos g\phi_0 = \cos g\phi_0$. This implies that $\cos g\phi_0 = 0$, hence $\phi_0 = \frac{\pi}{2g}$.

(ii) For $g = 4$, $y_2(t) = y_1^2(t) \cos 4\phi_0$ implies that

$$y_2' = 2y_1 y_1' \cos 4\phi_0 = -4ny_1 \cos 4\phi_0 = -8(m_2 - m_1)y_1.$$

Hence $\cos 4\phi_0 = \frac{2(m_2 - m_1)}{n} = \frac{m_2 - m_1}{m_1 + m_2}$. This proves the claim.

Next we compute the maximal time T for the MCF for the above examples. The flow blows up when $\cos g\phi(t) = \pm 1$. If $g = 3$ or 6 , then

$$y_2(t) = r(t)^g \cos g\phi(t) = (1 - 2nt)^{g/2} \cos g\phi(t) = \cos g\phi_0.$$

Hence $T = \frac{1}{2n}(1 - |\cos g\phi_0|)^{2/g}$. For $g = 4$, we have

$$y_2(t) = r(t)^4 \cos 4\phi(t) = (1 - 2nt)^2 \cos 4\phi(t) = \cos 4\phi_0 - 8(m_2 - m_1)(t - nt^2).$$

Then $\cos g\phi(T) = 1$ if $\phi_0 \in (0, \phi_4)$, and $\cos g\phi(T) = -1$ if $\phi_0 \in (\phi_4, \frac{\pi}{4})$. This solves T and we get:

Proposition 4.5. *Let ϕ_g be the constant defined by (4.14). Then*

(1) *The MCF through $e^{i\phi_g}$ homothetically converges to 0 at time $T = \frac{1}{2n}$.*

(2) *For $g = 3, 6$, the maximal time for the MCF with initial data $e^{i\phi_0}$ is*

$T = \frac{1}{2n}(1 - |\cos g\phi_0|^{\frac{2}{g}})$. For $g = 4$, the maximal time is

$$T = \begin{cases} \frac{1}{2n} \left(1 - \sqrt{\frac{m_1 + m_2}{2m_1} (\cos 4\phi_0 - \frac{m_2 - m_1}{m_2 + m_1})} \right), & \text{if } \theta_0 \in (0, \theta_4), \\ \frac{1}{2n} \left(1 - \sqrt{\frac{m_1 + m_2}{2m_2} (-\cos 4\phi_0 + \frac{m_2 - m_1}{m_2 + m_1})} \right), & \text{if } \theta_0 \in (\theta_4, \frac{\pi}{4}). \end{cases}$$

(3)

$$\lim_{t \rightarrow T^-} \phi(t) = \begin{cases} 0, & \text{if } \phi_0 \in (0, \phi_g), \\ \frac{\pi}{g}, & \text{if } \phi_0 \in (\phi_g, \frac{\pi}{g}). \end{cases}$$

5. MEAN CURVATURE FLOWS OF FOCAL SUBMANIFOLDS

We consider the mean curvature flows of focal submanifolds of isoparametric submanifolds. They behave very similarly to the mean curvature flows of isoparametric submanifolds. With slight modification, most results in section 3 also hold for mean curvature flows of focal submanifolds.

Let M_0 be an isoparametric submanifold and $p \in M_0$ a fixed point. As before, we assume that M_0 is contained in a sphere centered at the origin. Let $C \subset p + \nu_p M_0$ be the Weyl chamber containing p , and Δ^+ the set of positive roots of the Weyl group W associated to M_0 . Then

- (1) \bar{C} is a stratified space,
- (2) the isotropy subgroup of any two points in the same stratum σ are the same, and will be denoted by W_σ ,

(3) for $x \in \partial C$, let

$$\Delta^+(x) = \{\alpha \in \Delta^+ \mid \langle x, \alpha \rangle > 0\},$$

then $\Delta^+(x_1) = \Delta^+(x_2)$ if and only if x_1, x_2 lie in the same stratum σ , and will be denoted by $\Delta^+(\sigma)$,

(4) σ is the Weyl chamber of W_x for $x \in \sigma$ and σ is an open simplicial cone in the following linear subspace

$$V(\sigma) = \{x \in p + \nu_p M_0 \mid \langle x, \alpha \rangle = 0, \text{ for all } \alpha \in \Delta^+ \setminus \Delta^+(\sigma)\},$$

Let σ be a stratum in ∂C , $x_0 \in \sigma$, and M the focal submanifold of M_0 through x_0 . By [T, Theorem 4.1], the mean curvature vector field of M at x_0 is given by

$$H(x_0) = - \sum_{\alpha \in \Delta^+(\sigma)} \frac{m_\alpha}{\langle x_0, \alpha \rangle} \alpha \quad (5.1)$$

where m_α are multiplicities of curvature spheres of M_0 . Moreover

$$\langle x_0, \alpha \rangle = \langle H(x_0), \alpha \rangle = 0 \quad (5.2)$$

for all $\alpha \in \Delta^+ \setminus \Delta^+(\sigma)$. The mean curvature flow equation of M is the following ODE on σ :

$$\frac{dx}{dt} = - \sum_{\alpha \in \Delta^+(\sigma)} \frac{m_\alpha}{\langle x, \alpha \rangle} \alpha. \quad (5.3)$$

The analogue of Theorem 3.2 also holds for this case. In particular, if $x(t)$ satisfies the flow equation (5.3) then

$$\|x(t)\|^2 = \|x(0)\|^2 - 2nt \quad (5.4)$$

where $n = \sum_{\alpha \in \Delta^+(\sigma)} m_\alpha$ is the dimension of M . Therefore we have

Theorem 5.1. *The maximal interval for the solution of the mean curvature flow equation for any focal submanifold is finite.*

Suppose $x(t)$ and $y(t)$ satisfy equation (5.3) on σ and $x(0) \neq y(0)$. Use the same computation for (3.4) to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|x(t) - y(t)\|^2 &= \langle x(t) - y(t), x'(t) - y'(t) \rangle \\ &= \sum_{\alpha \in \Delta^+(\sigma)} m_\alpha \frac{\langle x(t) - y(t), \alpha \rangle^2}{\langle x(t), \alpha \rangle \langle y(t), \alpha \rangle} > 0. \end{aligned} \quad (5.5)$$

Then proofs given in section 3 works, so we have

Theorem 5.2. *Let $M^n \subset S^{n+k-1}$ be a compact isoparametric submanifold in \mathbb{R}^{n+k} , W its Weyl group, C the Weyl chamber in $x_0 + \nu(M)_{x_0}$ containing $x_0 \in M$, and M_y the submanifold parallel to M through y . If $\sigma \subset \overline{C}$ is a stratum, then*

- (1) *there is a unique $y_\sigma \in \sigma$ such that the focal submanifold M_{y_σ} is minimal in S^{n+k-1} , and the MCF in \mathbb{R}^{n+k} with initial data M_{y_σ} homothetically shrinks to a point,*

- (2) if $y_0 \in \sigma \cap S^{n+k-1} - \{y_\sigma\}$, then the MCF in \mathbb{R}^{n+k} with M_{y_0} as initial data blows up in finite time $T < \frac{1}{2n}$, $x(t) \in \sigma$ for all $t \in [0, T)$, and $\lim_{t \rightarrow T^-} x(t) \in \partial\sigma$, in particular, the limit is a focal submanifold with lower dimension,
- (3) if $y_1, y_2 \in \sigma \cap S^{n+k-1}$ are distinct, then the MCF in \mathbb{R}^{n+k} with initial data M_{y_1} and M_{y_2} converge to distinct focal submanifolds of lower dimensions.

6. MEAN CURVATURE FLOWS FOR ISOPARAMETRIC SUBMANIFOLDS IN SPHERES

If $M^n \subset S^{n+k-1}$ is an isoparametric submanifold in \mathbb{R}^{n+k} , then M is also isoparametric in \mathbb{R}^{n+k} . So basic structure theory for isoparametric submanifolds in Euclidean spaces also applies to M . For $x \in M$, let $H(x)$ and $H_E(x)$ be the mean curvature vector fields of M at x as a submanifold of S^{n+k-1} and \mathbb{R}^{n+k} respectively. Then $H(x)$ is the orthogonal projection of $H_E(x)$ to $T_x S^{n+k-1}$. More precisely

$$H(x) = H_E(x) + nx$$

for all $x \in M$. In particular, H is again a parallel normal vector field along M . The mean curvature flow of M as a submanifold of S^{n+k-1} behaves similarly to its flow as a submanifold of \mathbb{R}^{n+k} . With slight modifications, most results for mean curvature flows for isoparametric submanifolds in the Euclidean spaces also hold for isoparametric submanifolds in spheres. We only need to explain how to deal with the arguments in the Euclidean case which can not be applied directly to the spherical case.

Fix $x_0 \in M$ and let $V = x_0 + \nu_{x_0}M$ be the normal space of M as a submanifold of \mathbb{R}^N at the point x_0 , W its Coxeter group, and $C \subset V$ the Weyl chamber containing x_0 . The mean curvature flow of M in S^{n+k-1} is uniquely determined by the flow of x_0 in $S := C \cap S^{k-1}$:

$$x'(t) = - \sum_{\alpha \in \Delta_+} \frac{m_\alpha \alpha}{\langle x(t), \alpha \rangle} + nx(t). \quad (6.1)$$

The set S is a geodesic $(k-1)$ -simplex on S^{k-1} . Let $x(t) \in S$ be a solution to equation (6.1) with initial condition x_0 . Then

$$y(t) = \sqrt{1-2nt} \ x \left(-\frac{1}{2n} \log(1-2nt) \right)$$

satisfies the Euclidean mean curvature flow equation (4.1) with initial condition $y(0) = x_0$. Let $[0, T_x)$ and $[0, T_y)$ be the maximal intervals for the domains of $x(t)$ and $y(t)$ respectively. Then

$$T_x = -\frac{1}{2n} \log(1-2nT_y)$$

and

$$\lim_{t \rightarrow T_y^-} y(t) = \sqrt{1 - 2nT_y} \lim_{t \rightarrow T_x^-} x(t).$$

Note that by Theorems 3.2 and Corollary 3.9, $T_y \leq \frac{1}{2n}$ and the equality holds if and only if the isoparametric submanifold M_0 passing x_0 is minimal in the sphere S^{n+k-1} . So by Theorem 3.5, if M_0 is not minimal in the sphere, then $x(t)$ converges to a focal submanifold at a finite time T_x . This proves Theorem 1.2.

If $x_1(t) \in S$ and $x_2(t) \in S$ satisfy the spherical mean curvature flow equation (6.1), then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|x_1(t) - x_2(t)\|^2 &= \langle x_1 - x_2, (H_E(x_1) + nx_1) - (H_E(x_2) + nx_2) \rangle \\ &= n \|x_1 - x_2\|^2 + \langle x_1 - x_2, H_E(x_1) - H_E(x_2) \rangle. \end{aligned}$$

By (3.4), $\langle x_1 - x_2, H_E(x_1) - H_E(x_2) \rangle \geq 0$. Therefore

$$\frac{d}{dt} \|x_1(t) - x_2(t)\|^2 \geq 2n \|x_1(t) - x_2(t)\|^2. \quad (6.2)$$

We use (6.2) to give an estimate of the maximal interval $[0, T)$ for the spherical mean curvature flow $x(t)$. Let p_0 be the unique point in S such that the isoparametric submanifold passing p_0 is minimal in the sphere S^{n+k-1} . Set $x_1(t) = x(t)$ and $x_2(t) = p_0$ in equation (6.2). Since $x_2(t)$ exists for all $t > 0$, we obtain

$$\|x(t) - p_0\| \geq e^{nt} \|x(0) - p_0\|$$

for all t as long as $x(t) \in S$. Let D be the diameter of S , then $D \leq 2$ and

$$T \leq \frac{1}{n} \log \frac{D}{\|x(0) - p_0\|}.$$

Now we discuss the behavior of invariant polynomials under the spherical mean curvature flow. Let $x(t) \in S$ be the mean curvature flow of x_0 . For any function f on V , let $f(t) = f(x(t))$. Then

$$f'(t) = \langle \nabla f(x(t)), H(x(t)) \rangle = \langle \nabla f(x(t)), H_E(x(t)) \rangle + n \langle \nabla f(x(t)), x(t) \rangle.$$

If f is a homogenous polynomial of degree k which is invariant under the action of the Coxeter group W , then as in the proof of Theorem 3.5,

$$f'(t) = -F(x(t)) + nk f(t) \quad (6.3)$$

where F is defined by equation (2.4) and it is an invariant polynomial of degree $k - 2$. If we have computed $F(t) := F(x(t))$, then we can solve $f(t)$ from equation (6.3) and obtain

$$f(t) = -e^{knt} \int e^{-knt} F(t) dt. \quad (6.4)$$

Note that there is no homogeneous invariant polynomial of degree 1. By induction on the degree, we obtain the following

Theorem 6.1. *If $x(t)$ satisfies the spherical mean curvature flow equation (6.1) and f is a W -invariant polynomial, then $f(t) = f(x(t)) = c_1 e^{knt} + h(t)$ for some constant c_1 and polynomial h .*

In particular $f(t)$ is well defined for all $t \in \mathbb{R}$. In section 4, we have given explicit formulas for F_i for invariant homogeneous polynomials P_i for isoparametric submanifolds. We can use these formula and (6.4) to construct explicit solutions to the spherical mean curvature flow equation for isoparametric submanifolds in spheres.

Example 6.2. Phase portrait for rank 2 cases

Let $M^n \subset S^{n+1} \subset \mathbb{R}^{n+2}$ be an isoparametric hypersurface with g distinct principal curvatures. Then the Weyl group associated to M as a rank 2 isoparametric submanifold in \mathbb{R}^{n+2} is the dihedral group of $2g$ elements. Let C denote the Weyl chamber containing $x_0 \in M$, and D the intersection of C and the normal circle at x_0 in S^{n+1} . Let p_1, p_2 denote the end points of D . The arc $D = \widehat{p_1 p_2}$ has length π/g . For $y \in \bar{C}$, let M_y denote the submanifold through y that is parallel to M (a leaf of the isoparametric foliation). There exists a unique $p_0 \in D$ such that M_{p_0} is minimal in S^{n+1} .

- (1) The spherical MCF (6.1) has three orbits: a stationary point p_0 , the orbit $\widehat{p_0 p_1}$ with one end tends to p_0 and the other end tends to p_1 , and the orbit $\widehat{p_0 p_2}$ with one end tends to p_0 and the other end tends to p_2 .
- (2) The MCF (3.2) in \mathbb{R}^{n+2} starting at M_y degenerates homothetically to one point (the origin) if $y = p_0$, to M_{rp_2} for some $0 < r < 1$ if $y \in \widehat{p_0 p_2}$, and to M_{rp_1} for some $1 < r < 1$ if $y \in \widehat{p_1 p_0}$.

Example 6.3. Phase portrait for the A_3 cases

Let $M^n \subset S^{n+2}$ be an isoparametric submanifold with Weyl group A_3 and uniform multiplicity m , and $x_0 \in M$. Let C denote the Weyl chamber containing x_0 , and D the intersection of C and the normal sphere at x_0 . Then D is a geodesic triangle with vertices p_1, p_2, p_3 and interior angles $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2}$. The phase spaces of spherical MCF (6.1) and Euclidean MCF (3.2) are D and C respectively. We describe the phase portraits:

(1) There exists a unique p_0 in the interior of D such that p_0 is the fixed point of the ODE (6.1). This implies that the spherical MCF starting at the parallel submanifold M_{p_0} is stationary and M_{p_0} is minimal in S^{n+k-1} .

(2) For each $1 \leq i \leq 3$, there is a unique flow ℓ_i that at one end approaches p_0 and at the other end approaches p_i . This implies that for $y \in \ell_i$, the spherical MCF starting at M_y collapses in finite time to the focal submanifold M_{p_i} by collapsing fibers of the fibration $M_y \rightarrow M_{p_i}$. The fibers are isoparametric submanifolds with Weyl group A_2 for $i = 1, 2$ and $A_1 \times A_1$ for $i = 3$. In particular, when $m = 2$, M_y is diffeomorphic to the manifold of flags in \mathbb{C}^4 and collapsing is along complex flag manifolds of \mathbb{C}^3 in M_y for $i = 1, 2$ and $S^2 \times S^2$ in M_y for $i = 3$.

(3) For distinct i, j, k , let D_k denote the triangle with vertices p_i, p_j, p_0 and edges ℓ_i, ℓ_j , and geodesic segment $\widehat{p_i p_j}$ in the sphere. The flow for (6.1) starting at a point in the interior D_k^0 of D_k exists for finite time and converges to a point on the interior of $\widehat{p_i p_j}$. This implies that for $y \in D_k^0$, the spherical MCF starting at M_y converges in finite time to a focal submanifold M_q with $q \in \widehat{p_i p_j} \setminus \{p_i, p_j\}$ by collapsing one family of curvature spheres.

(4) The flow of (3.2) on C starting at p_0 is the straight line joining the origin to p_0 , the flow starting from a point in D_k^0 converges to a point on the wall containing p_i, p_j (i, j, k distinct), and the flow starting at a point on ℓ_i converges to a point on the line segment $\overline{Op_i}$. This implies that the Euclidean MCF with initial data M_y

- (i) shrinks homothetically to the origin if $y = p_0$,
- (ii) converges to a focal submanifold M_q for some q lies in the open cone spanned by $\widehat{p_i p_j}$ and the collapsing is along a curvature m -sphere if $y \in D_k^0$,
- (iii) converges to a focal submanifold M_q with $q \in \overline{Op_i}$ for $y \in \ell_i$, moreover, the collapsing is along fibers of the fibration $M_y \rightarrow M_q$ and the fibers are isoparametric submanifolds with Weyl group A_2, A_2 , and $A_1 \times A_1$ respectively for $i = 1, 2, 3$.

7. MEAN CURVATURE FLOW FOR POLAR ACTION ORBITS

Let G act on a Riemannian manifold N isometrically, and $G \cdot p$ be a principal orbit through p . If $v \in \nu(G \cdot p)_p$, then $\hat{v}(g \cdot p) = dg_p(v)$ is a globally defined normal vector field on $G \cdot p$ and is called a G -equivariant normal field. An isometric action of a compact Lie group G on a Riemannian manifold N is called *polar* if there is a totally geodesic submanifold Σ that meets all G -orbits and meets orthogonally. Such Σ is called a *section*. We list some properties of polar actions (cf. [PT]):

- (1) The action G is polar if and only if every G -equivariant normal field is parallel with respect to the induced normal connection on $G \cdot p$ as a submanifold of N .
- (2) Let $N(\Sigma) = \{g \in G \mid g \cdot \Sigma = \Sigma\}$ and $Z(\Sigma) = \{g \in G \mid g \cdot x = x \ \forall x \in \Sigma\}$ denote the normalizer and centralizer of Σ respectively. Then the quotient group $W(\Sigma) = N(\Sigma)/Z(\Sigma)$ is a finite group acting on Σ , and is called *the generalized Weyl group associated to the polar action*.
- (3) The orbit space Σ/W is isomorphic to N/G and the ring of smooth G -invariant functions on N is isomorphic to the ring of W -invariant functions on Σ under the restriction map.
- (4) If $p_0 \in \Sigma$ is a singular point, i.e., $G \cdot p_0$ is a singular orbit, then the slice representation of G_{p_0} on the normal space of the orbit at p_0 is a polar representation.

Theorem 7.1. *Suppose the isometric action G on N is polar, and Σ is a section. Then*

- (i) *if $x \in \Sigma$, then the mean curvature vector $\xi(x)$ of $G \cdot x$ at x is tangent to Σ at x ,*
- (ii) *if $x'(t) = \xi(x(t))$ with $x(t)$ regular (i.e., $G \cdot x(t)$ is a principal orbit), then $G \cdot x(t)$ satisfies the MCF in N , in other words, the MCF in N with a principal G -orbit as initial data flows among principal G -orbits.*

For general polar action, W need not be a Coxeter group and the orbit space of the W -action on the section can be complicated. In fact, given any finite group W and any compact W -manifold, there exist a Riemannian manifold N , a compact group G , and an isometric polar G -action on N such that the induced action on the section is the given W -action (cf. [PT]). Hence the behavior of the MCF for general polar actions is not as clear as in the sphere and Euclidean case.

A polar action on a symmetric space is *hyperpolar* if the sections are flat. In this case the fundamental domain of the W -action on a section is a geodesic simplex. A submanifold in a symmetric space is called *equifocal* if its normal bundle is flat, exponential of each normal space is a flat, and the focal radii along a parallel normal field are constant. It was proved in [TT] that principal orbits of a hyperpolar action on symmetric space are equifocal, and parallel foliation of an equifocal submanifold is an orbit-like foliation. Moreover, generators of the ring of smooth functions that are constant along parallel leaves were constructed in [HLO]. Hence we believe that methods developed in this paper can be used to solve the MCF starting with an equifocal submanifold in symmetric spaces.

REFERENCES

- [FKM] D. Ferus, H. Karcher, and H.F. Münzner, *Cliffordalgebren und neue isoparametrische hyperflächen*, Math. Z. 177 (1981), 479-502.
- [GH] M.E. Gage and R.S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geometry 23 (1986), 69-96.
- [GB] L.C. Grove and C.T. Benson, *Finite reflection groups*, second edition, GTM 99, Springer-Verlag, 1985.
- [HLO] E. Heintze, X. Liu, C. Olmos, *Isoparametric submanifolds and a Chevalley-type restriction theorem*, Integrable systems, geometry, and topology, AMS/IP Stud. Adv. Math., 36 (2006), 151-190.
- [HOT] E. Heintze, C. Olmos, and G. Thorbergsson, *Submanifolds with constant principal curvatures and normal holonomy group*, Int. J. Math. 2 (1991), 167-175.
- [Hu] G. Huisken, *Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature*, Invent. Math. 84 (1986), 463-480.
- [PT] R.S. Palais, C.-L. Terng, *A general theory of canonical forms*, Transaction, Amer. Math. Soc. 300 (1987) 771-789.
- [PTb] R.S. Palais, C.-L. Terng, *Critical Point Theory and Submanifold Geometry*, Lecture Notes in Math., vol. **1353** (1988), Springer-Verlag
- [T] C.-L. Terng, *Isoparametric submanifolds and their Coxeter groups*, J. Differential Geometry, 21 (1985), 79-107.

- [TT] C.-L. Terng, G. Thorbergsson, *Submanifold Geometry in Symmetric spaces*, J. Differential Geometry, 42 (1995) 665-718.
- [Th] G. Thorbergsson, *Isoparametric foliations and their buildings*, Ann. of Math., 133 (1991), 429-446.
- [V] V.S. Varadarajan, *Lie groups, Lie algebras, and their representations*, GTM 102, Springer-Verlag, 1984.
- [W] M.-T. Wang, *Mean curvature flow in higher codimension*, to appear in the Proceedings of International Congress of Chinese Mathematicians 2001, math.DG/0204054
- [Z] X.-P. Zhu, *Lectures on mean curvature flows*, Studies in Advanced Math., AMS/IP, 2002.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46566

E-mail address: xliu3@nd.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT IRVINE, IRVINE, CA 92697-3875

E-mail address: cterng@math.uci.edu